

# $CP^1$ model with Hopf term and fractional spin statistics

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## Abstract

We reconsider the  $CP^1$  model with the Hopf term by using the Batalin-Fradkin-Tyutin (BFT) scheme, which is an improved version of the Dirac quantization method. We also perform a semi-classical quantization of the topological charge  $Q$  sector by exploiting the collective coordinates to explicitly show the fractional spin statistics.

Keywords:  $CP^1$  model, Hopf term, BFT formalism

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## I. INTRODUCTION

Since the (2+1) dimensional  $O(3)$  nonlinear sigma model (NLSM) was first discussed by Belavin and Polyakov [1], there have been lots of attempts to improve this soliton model associated with the homotopy group  $\pi_2(S^2) = \mathbb{Z}$ . In particular, the configuration space in the  $O(3)$  NLSM is infinitely connected to yield the fractional spin statistics, which was first shown by Wilczek and Zee [2] via the additional Hopf term. The creation and annihilation mechanism of a Skyrmion-anti Skyrmion pair in the vacuum through the channel of  $2\pi$  rotation of the Skyrmion was also studied in the  $O(3)$  NLSM [3] to discuss the Hopf topological invariant and linking number [2]. Moreover the  $O(3)$  NLSM with the Hopf term was canonically quantized [4] and the  $CP^1$  model with the Hopf term [5–9], which can be related with the  $O(3)$  NLSM via the Hopf map projection from  $S^3$  to  $S^2$ , was also canonically quantized later [6]. In fact, the  $CP^1$  model has better features than the  $O(3)$  NLSM, in the sense that the action of the  $CP^1$  model with the Hopf invariant has a desirable manifest locality, since the Hopf term has a local integral representation in terms of the physical fields of the  $CP^1$  model [2]. Furthermore, this manifest locality in time is crucial for a consistent canonical quantization. However, there still exist several ambiguities in performing the quantization rigorously and in deriving explicit expressions of fractional spin. [5–9]

On the other hand, physical systems with constraints was systematically studied by Dirac [10], according to whom in the second class constraint system one needs to use the Dirac brackets instead of the Poisson brackets to proceed to quantize the physical system. However, in this Dirac quantization scheme, we have difficulties in finding canonically conjugate pairs due to field operator ordering ambiguity. To circumvent such problems, Batalin, Fradkin and Tyutin (BFT) [11–13] invented a scheme which converts the second class constraints into first class ones by introducing auxiliary fields. Recently this BFT scheme has been applied to several areas of current interests such as the soliton models [14,15], high dense matter physics [16] and D-brane systems [17].

The motivation of this paper is to systematically apply the BFT scheme [11–13], which is

an improved version of the Dirac quantization method, the Batalin, Fradkin and Vilkovisky (BFV) method [18] and the Becci-Rouet-Stora-Tyutin (BRST) method [19] to the  $CP^1$  model with the Hopf term [5–9]. As a result, we will explicitly show that the  $CP^1$  model has a fractional spin statistics. In section 2 we convert the second-class constraints into first-class ones following the BFT method to construct first-class BFT physical fields and directly derive the compact expression of a first-class Hamiltonian in terms of these fields. We construct in section 3 a BRST-invariant gauge fixed Lagrangian in the BFV scheme through the standard path-integral procedure. Exploiting collective coordinates, in section 4 we perform a semi-classical quantization to describe the fractional statistics explicitly.

## II. FIRST-CLASS CONSTRAINTS AND FIRST-CLASS HAMILTONIAN

In this section, let us apply the BFT scheme to the (2+1) dimensional  $CP^1$  model with a Hopf term [5–7], which is a second-class constraint system, and whose Lagrangian is given as

$$L = \int d^2x \left[ (D_\mu Z_\alpha)^* (D^\mu Z_\alpha) + \frac{\Theta}{4\pi^2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right] \quad (2.1)$$

with  $D_\mu = \partial_\mu - iA_\mu$  and  $A_\mu$  is defined as

$$A_\mu = -\frac{i}{2} (Z_\alpha^* \partial_\mu Z_\alpha - Z_\alpha \partial_\mu Z_\alpha^*). \quad (2.2)$$

Here  $Z_\alpha = (Z_1, Z_2)$  is a multiplet of complex scalar fields with a constraint

$$\Omega_1 = Z_\alpha^* Z_\alpha - 1 = |Z|^2 - 1 \approx 0. \quad (2.3)$$

On the other hand, this Lagrangian (2.1) can also be rewritten as the following standard form [8],

$$\begin{aligned} L &= \int d^2x \left[ \partial_\mu Z_\alpha^* \partial^\mu Z_\alpha - (Z_\alpha^* \partial_\mu Z_\alpha) (Z_\beta \partial^\mu Z_\beta^*) + \lambda \Omega_1 + \mathcal{L}_H \right], \\ \mathcal{L}_H &= -\frac{\Theta}{8\pi^2} \epsilon^{\mu\nu\rho} (Z_\alpha^* \partial_\mu Z_\alpha - \partial_\mu Z_\alpha^* Z_\alpha) \partial_\nu Z_\beta^* \partial_\rho Z_\beta. \end{aligned} \quad (2.4)$$

This Lagrangian is invariant under a local U(1) gauge symmetry transformation

$$Z_\alpha(x) \rightarrow e^{i\theta(x)} Z_\alpha(x), \quad (2.5)$$

and we have explicitly included the constraint  $\Omega_1$ . By performing the Legendre transformation, one can obtain the canonical Hamiltonian,

$$\begin{aligned} H_c &= H - \int d^2x \lambda \Omega_1 \\ H &= \int d^2x \left[ |\Pi_\alpha + \frac{\Theta}{8\pi^2} \Pi_\alpha^H|^2 + |\partial_i Z_\alpha|^2 - (Z_\alpha^* \partial_i Z_\alpha)(Z_\beta \partial_i Z_\beta^*) \right] \end{aligned} \quad (2.6)$$

where  $\Pi_\alpha$  are the canonical momenta conjugate to the complex scalar fields  $Z_\alpha$  given by

$$\begin{aligned} \Pi_\alpha &= \dot{Z}_\alpha^* - Z_\alpha^* Z_\beta \dot{Z}_\beta^* - \frac{\Theta}{8\pi^2} \Pi_\alpha^H, \\ \Pi_\alpha^H &= \epsilon^{ij} (Z_\alpha^* \partial_i Z_\beta^* \partial_j Z_\beta + (Z_\beta^* \partial_i Z_\beta - \partial_i Z_\beta^* Z_\beta) \partial_j Z_\alpha^*), \end{aligned} \quad (2.7)$$

and  $\Pi_\alpha^*$  are the complex conjugate of  $\Pi_\alpha$ . Even though there is no explicit physical contribution of the  $\Omega_1$  term to  $H_c$ , we have the canonical momentum conjugate to the multiplier field  $\lambda$  to yield the first class constraint  $\Omega_0 = \Pi_\lambda \approx 0$ . The time evolution of the constraint  $\Omega_0$  with  $H_c$  yields the constraint  $\Omega_1$  in Eq. (2.3), and subsequent time evolution of the constraint  $\Omega_1$  yields an additional secondary constraint

$$\Omega_2 = Z_\alpha^* (\Pi_\alpha^* + \frac{\Theta}{8\pi^2} \Pi_\alpha^{H*}) + Z_\alpha (\Pi_\alpha + \frac{\Theta}{8\pi^2} \Pi_\alpha^H) \approx 0. \quad (2.8)$$

On the other hand, we could fix the multiplier field  $\lambda$  such as  $\lambda = -\frac{1}{\Delta_{12}} \{\Omega_2, H\}$  so that we can have closed constraint algebra with no more time evolution of the constraints [10]. Since this multiplier field  $\lambda$  is non-dynamical after fixing as above, one can decouple the irrelevant conjugate pair  $(\lambda, \Pi_\lambda)$  from our system of interest. As a result, all the second-class constraints  $\Omega_1$  and  $\Omega_2$  form the following constraint algebra

$$\Delta_{kk'}(x, y) = \{\Omega_k(x), \Omega_{k'}(y)\} = 2\epsilon^{kk'} |Z|^2 \delta(x - y) \quad (2.9)$$

with  $\epsilon^{12} = -\epsilon^{21} = 1$ .

Next, we consider the Poisson brackets of the fields to construct the Dirac brackets defined as

$$\{A(x), B(y)\}_D = \{A(x), B(y)\} - \int d^2z d^2z' \{A(x), \Omega_k(z)\} \Delta^{kk'} \{\Omega_{k'}(z'), B(y)\} \quad (2.10)$$

with  $\Delta^{kk'}$  being the inverse of  $\Delta_{kk'}$  in Eq. (2.9). After some algebraic manipulation, we obtain the commutators as follows

$$\begin{aligned} \{Z_\alpha(x), Z_\beta(y)\}_D &= \{Z_\alpha^*(x), Z_\beta(y)\}_D = 0, \\ \{Z_\alpha(x), \Pi_\beta(y)\}_D &= (\delta_{\alpha\beta} - \frac{Z_\alpha Z_\beta^*}{2|Z|^2}) \delta(x-y), \\ \{Z_\alpha(x), \Pi_\beta^*(y)\}_D &= -\frac{Z_\alpha Z_\beta}{2|Z|^2} \delta(x-y), \\ \{\Pi_\alpha(x), \Pi_\beta(y)\}_D &= \frac{1}{2|Z|^2} (\Pi_\alpha Z_\beta^* - Z_\alpha^* \Pi_\beta - \frac{\Theta}{4\pi^2} (\Pi_\alpha^H Z_\beta^* - Z_\alpha^* \Pi_\beta^H)) \delta(x-y), \\ \{\Pi_\alpha(x), \Pi_\beta^*(y)\}_D &= \frac{1}{2|Z|^2} (\Pi_\alpha Z_\beta - Z_\alpha^* \Pi_\beta^* - \frac{\Theta}{4\pi^2} (\Pi_\alpha^H Z_\beta - Z_\alpha^* \Pi_\beta^{H*})) \delta(x-y). \end{aligned} \quad (2.11)$$

Note that we have the Hopf term contributions in the last two Dirac commutators in Eq. (2.11).

Now, we calculate the symmetric energy-momentum tensor

$$\begin{aligned} T^{\mu\nu} &= \partial^\mu Z^* \partial^\nu Z - (Z \partial^\mu Z^*) (Z^* \partial^\nu Z) - \frac{\Theta}{8\pi^2} \epsilon^{\mu\rho\sigma} ((Z^* \partial^\nu Z) (\partial_\rho Z^* \partial_\sigma Z) \\ &\quad + (Z^* \partial_\rho Z - \partial_\rho Z^* Z) (\partial_\sigma Z^* \partial^\nu Z)) + \text{c.c} \\ &\quad - g^{\mu\nu} (\partial_\sigma Z^* \partial^\sigma Z) + g^{\mu\nu} (Z^* \partial_\sigma Z) (Z \partial^\sigma Z^*) \\ &\quad + \frac{\Theta}{8\pi^2} g^{\mu\nu} \epsilon^{\alpha\beta\rho} (Z^* \partial_\alpha Z - \partial_\alpha Z^* Z) (\partial_\beta Z^* \partial_\rho Z), \end{aligned} \quad (2.12)$$

from which we can obtain the momentum operator  $P^i$  as

$$P^i = \int d^2x T^{0i} = \int d^2x (\Pi_\alpha \partial^i Z_\alpha + \Pi_\alpha^* \partial^i Z_\alpha^*). \quad (2.13)$$

This momentum operator  $P^i$  generates the desired translation as follows

$$\{P^i, Z_\alpha(x)\}_D = \partial^i Z_\alpha(x). \quad (2.14)$$

On the other hand, since the angular momentum operator  $J$  is given by

$$J = \int d^2x \epsilon_{ij} x^i T^{0j}, \quad (2.15)$$

the rotational property of the  $Z_\alpha$  field is obtained by treating the Dirac commutator

$$\{J, Z_\alpha(x)\}_D = \epsilon^{ij} x^i \partial_j Z_\alpha(x), \quad (2.16)$$

which shows that there is no anomaly term, contrast to the result of Ref. [6].

Following the BFT formalism [11–13], which systematically converts the second-class constraints into first-class ones, let us introduce two real auxiliary fields  $\Phi^i$  with the Poisson brackets

$$\{\Phi^i(x), \Phi^j(y)\} = \epsilon^{ij} \delta(x - y),$$

to obtain the first-class constraints as follows

$$\begin{aligned} \tilde{\Omega}_1 &= \Omega_1 + 2\Phi^1, \\ \tilde{\Omega}_2 &= \Omega_2 - |Z|^2 \Phi^2, \end{aligned} \quad (2.17)$$

which yield a strongly involutive first-class constraint algebra  $\{\tilde{\Omega}_i(x), \tilde{\Omega}_j(y)\} = 0$ . Here one notes that the physical fields  $Z_\alpha$  are geometrically constrained to reside on the  $S^3$  hypersphere with the modified norm  $|Z|^2 = 1 - 2\Phi^1$ .

Now, we construct the first class BFT physical fields  $\tilde{\mathcal{F}} = (\tilde{Z}_\alpha, \tilde{\Pi}_\alpha)$  corresponding to the original fields  $\mathcal{F} = (Z_\alpha, \Pi_\alpha)$ . These fields  $\tilde{\mathcal{F}}$ 's are obtained as a power series in the auxiliary fields  $\Phi^i$  by demanding that they are strongly involutive:  $\{\tilde{\Omega}_i, \tilde{\mathcal{F}}\} = 0$ . After some algebra, we obtain the compact forms of first class physical fields as

$$\begin{aligned} \tilde{Z}_\alpha &= Z_\alpha \left( \frac{|Z|^2 + 2\Phi^1}{|Z|^2} \right)^{1/2}, \\ \tilde{\Pi}_\alpha &= \left( \Pi_\alpha - \frac{1}{2} Z_\alpha^* \Phi^2 \right) \left( \frac{|Z|^2}{|Z|^2 + 2\Phi^1} \right)^{1/2}. \end{aligned} \quad (2.18)$$

As discussed in Ref. [13], any functional  $\mathcal{K}(\tilde{\mathcal{F}})$  of the first class fields  $\tilde{\mathcal{F}}$  is also first class, namely,  $\tilde{\mathcal{K}}(\mathcal{F}; \Phi) = \mathcal{K}(\tilde{\mathcal{F}})$ . Using this useful property, we easily construct a first-class Hamiltonian in terms of the above BFT physical variables omitting infinite iteration procedure to arrive at<sup>1</sup>

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<sup>1</sup>From now on, for simplicity we will ignore the term proportional to  $\Omega_1$ , which does not yield

$$\tilde{H} = \int dx \left[ |\tilde{\Pi}_\alpha + \frac{\Theta}{8\pi^2} \tilde{\Pi}_\alpha^H|^2 + |\partial_i \tilde{Z}_\alpha|^2 - (\tilde{Z}_\alpha^* \partial_i \tilde{Z}_\alpha) (\tilde{Z}_\beta \partial_i \tilde{Z}_\beta^*) \right]. \quad (2.19)$$

We then directly rewrite this Hamiltonian in terms of the original as well as auxiliary fields <sup>2</sup> to obtain

$$\begin{aligned} \tilde{H} = \int d^2x \left[ |\Pi_\alpha - \frac{1}{2} Z_\alpha^* \Phi^2 + \frac{\Theta}{8\pi^2 R^2} \Pi_\alpha^H|^2 R \right. \\ \left. + |\partial_i Z_\alpha|^2 \frac{1}{R} - (Z_\alpha^* \partial_i Z_\alpha) (Z_\beta \partial_i Z_\beta^*) \frac{1}{R^2} \right], \end{aligned} \quad (2.20)$$

where  $R = |Z|^2/(|Z|^2 + 2\Phi^1)$ . Here  $\tilde{H}$  is strongly involutive with the first class constraints  $\{\tilde{\Omega}_i, \tilde{H}\} = 0$ . A problem with  $\tilde{H}$  in (2.20) is that it does not naturally generate the first-class Gauss law constraint from the time evolution of the constraint  $\tilde{\Omega}_1$ . Therefore, by introducing an additional term proportional to the first class constraints  $\tilde{\Omega}_2$  into  $\tilde{H}$ , we obtain an equivalent first class Hamiltonian

$$\tilde{H}' = \tilde{H} + \frac{1}{2} \int d^2x \Phi^2 \tilde{\Omega}_2, \quad (2.21)$$

which naturally generates the form invariant Gauss law constraint

$$\{\tilde{\Omega}_1, \tilde{H}'\} = \tilde{\Omega}_2, \quad \{\tilde{\Omega}_2, \tilde{H}'\} = 0. \quad (2.22)$$

Note that  $\tilde{H}$  and  $\tilde{H}'$  act in the same way on physical states, which are annihilated by the first-class constraints.

### III. BRST SYMMETRIES

In this section we introduce two canonical sets of ghosts and anti-ghosts together with auxiliary fields in the framework of the BFV formalism [18], which is applicable to theories with the first-class constraints:

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any particular physical results.

<sup>2</sup>In deriving the first class Hamiltonian  $\tilde{H}$  of Eq. (2.20), we have used the conformal map condition,  $Z_\alpha^* \partial_i Z_\alpha + Z_\alpha \partial_i Z_\alpha^* = 0$ .

$$(\mathcal{C}^i, \bar{\mathcal{P}}_i), \quad (\mathcal{P}^i, \bar{\mathcal{C}}_i), \quad (N^i, B_i), \quad (i = 1, 2)$$

which satisfy the super-Poisson algebra <sup>3</sup>

$$\{\mathcal{C}^i(x), \bar{\mathcal{P}}_j(y)\} = \{\mathcal{P}^i(x), \bar{\mathcal{C}}_j(y)\} = \{N^i(x), B_j(y)\} = \delta_j^i \delta(x - y).$$

In the  $CP^1$  model, the nilpotent BRST charge  $Q_B$  and the BRST invariant minimal Hamiltonian  $H_m$  are given by

$$\begin{aligned} Q_B &= \int d^2x \, (\mathcal{C}^i \tilde{\Omega}_i + \mathcal{P}^i B_i), \\ H_m &= \tilde{H}' - \int d^2x \, \mathcal{C}^1 \bar{\mathcal{P}}_2, \end{aligned} \tag{3.1}$$

which satisfy the relations

$$\{Q_B, H_m\} = 0, \quad Q_B^2 = \{Q_B, Q_B\} = 0. \tag{3.2}$$

Our next task is to fix the gauge, which is crucial to identify the BFT auxiliary field  $\Phi^1$  with the Stüeckelberg field. The desired identification follows if one chooses the fermionic gauge fixing function  $\Psi$  as

$$\Psi = \int d^2x \, (\bar{\mathcal{C}}_i \chi^i + \bar{\mathcal{P}}_i N^i), \tag{3.3}$$

with the unitary gauge

$$\chi^1 = \Omega_1, \quad \chi^2 = \Omega_2. \tag{3.4}$$

Here note that the  $\Psi$  satisfies the following identity

$$\{\{\Psi, Q_B\}, Q_B\} = 0. \tag{3.5}$$

The effective quantum Lagrangian is then described as

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<sup>3</sup>Here the super-Poisson bracket is defined as  $\{A, B\} = \frac{\delta A}{\delta q}|_r \frac{\delta B}{\delta p}|_l - (-1)^{\eta_A \eta_B} \frac{\delta B}{\delta q}|_r \frac{\delta A}{\delta p}|_l$  where  $\eta_A$  denotes the number of fermions, called the ghost number, in  $A$  and the subscript  $r$  and  $l$  denote right and left derivatives, respectively.



$$L_{eff} = \int d^2x \left( \Pi_\alpha^* \dot{Z}_\alpha^* + \Pi_\alpha \dot{Z}_\alpha + \pi_\theta \dot{\theta} + B_2 \dot{N}^2 + \bar{\mathcal{P}}_i \dot{\mathcal{C}}^i + \bar{\mathcal{C}}_2 \dot{\mathcal{P}}^2 \right) - H_{tot} \quad (3.6)$$

with  $H_{tot} = H_m - \{Q_B, \Psi\}$ . We have identified here the auxiliary fields  $\Phi^i$  with a canonical conjugate pair  $(\theta, \pi_\theta)$ , namely,

$$\Phi^i = (\theta, \pi_\theta), \quad (3.7)$$

and the terms  $\int d^2x (B_1 \dot{N}^1 + \bar{\mathcal{C}}_1 \dot{\mathcal{P}}^1) = \{Q_B, \int d^2x \bar{\mathcal{C}}_1 \dot{N}^1\}$  have been suppressed by replacing  $\chi^1$  with  $\chi^1 + \dot{N}^1$ .

Now let us perform path integration over the fields  $B_1$ ,  $N^1$ ,  $\bar{\mathcal{C}}_1$ ,  $\mathcal{P}^1$ ,  $\bar{\mathcal{P}}_1$  and  $\mathcal{C}^1$ , by using the equations of motion. This leads to the effective Lagrangian of the form

$$\begin{aligned} L_{eff} = \int d^2x \left[ \Pi_\alpha^* \dot{Z}_\alpha^* + \Pi_\alpha \dot{Z}_\alpha + \pi_\theta \dot{\theta} + B \dot{N} + \bar{\mathcal{P}} \dot{\mathcal{C}} + \bar{\mathcal{C}} \dot{\mathcal{P}} \right. \\ \left. - |\Pi_\alpha - \frac{1}{2} Z_\alpha^* \Phi^2 + \frac{\Theta}{8\pi^2 R^2} \Pi_\alpha^H|^2 R + (Z_\alpha^* \partial_i Z_\alpha) (Z_\beta \partial_i Z_\beta^*) \frac{1}{R^2} \right. \\ \left. - |\partial_i Z_\alpha|^2 \frac{1}{R} - \frac{1}{2} \pi_\theta \tilde{\Omega}_2 + 2|Z|^2 \pi_\theta \bar{\mathcal{C}} \mathcal{C} + \tilde{\Omega}_2 N + B \Omega_2 + \bar{\mathcal{P}} \mathcal{P} \right] \end{aligned} \quad (3.8)$$

with the redefinitions:  $N \equiv N^2$ ,  $B \equiv B_2$ ,  $\bar{\mathcal{C}} \equiv \bar{\mathcal{C}}_2$ ,  $\mathcal{C} \equiv \mathcal{C}^2$ ,  $\bar{\mathcal{P}} \equiv \bar{\mathcal{P}}_2$ ,  $\mathcal{P} \equiv \mathcal{P}_2$ .

After performing the routine variation procedure and identifying  $N = -B + \dot{\theta}/(1 - 2\theta)$  we arrive at the effective Lagrangian of the covariant form

$$\begin{aligned} L_{eff} = \int d^2x \left[ \frac{1}{1 - 2\theta} (\partial_\mu Z_\alpha^*) (\partial^\mu Z_\alpha) - \frac{1}{(1 - 2\theta)^2} (Z_\alpha^* \partial_\mu Z_\alpha) (Z_\beta \partial^\mu Z_\beta^*) \right. \\ \left. + \frac{1}{(1 - 2\theta)^2} \mathcal{L}_H - (1 - 2\theta)^2 (B + 2\bar{\mathcal{C}} \mathcal{C})^2 - \frac{1}{1 - 2\theta} \partial_\mu \theta \partial^\mu B + \partial_\mu \bar{\mathcal{C}} \partial^\mu \mathcal{C} \right] \end{aligned} \quad (3.9)$$

which is invariant under the BRST-transformation

$$\begin{aligned} \delta_B Z_\alpha &= \lambda Z_\alpha \mathcal{C}, & \delta_B \theta &= -\lambda(1 - 2\theta) \mathcal{C}, \\ \delta_B \bar{\mathcal{C}} &= -\lambda B, & \delta_B \mathcal{C} &= \delta_B B = 0. \end{aligned} \quad (3.10)$$

Note that at the level of Lagrangian, if we take the limit of  $\theta \rightarrow 0$ , and integrate out decoupled auxiliary fields and (anti)ghost fields, one could directly recover the original unitary gauge-fixed Lagrangian (2.4).

#### IV. COLLECTIVE COORDINATE QUANTIZATION

In this section, we perform a semi-classical quantization of the topological charge  $Q$  sector of the  $CP^1$  model by exploiting the collective coordinates to consider physical aspects of the theory.

In order to consider the quantum ground state, let us explicitly treat zero modes responsible for classical degeneracy by introducing desired collective coordinates [6] satisfying the constraint  $|Z|^2 = 1$  as follows

$$\begin{aligned} Z_1 &= e^{i\alpha} \cos \frac{F(r)}{2}, \\ Z_2 &= e^{i\phi} \sin \frac{F(r)}{2}, \end{aligned} \quad (4.1)$$

where  $(r, \phi)$  are the polar coordinates and  $\alpha(t)$  is the collective coordinates. Here, we have used the profile function  $F(r)$ , which satisfies the boundary conditions:  $\lim_{r \rightarrow \infty} F(r) = \pi$  and  $F(0) = 0$ .

It seems appropriate to comment on the collective coordinate ansatz (4.1), which yields explicit contributions of the Hopf term to the physical quantities as below. In general, for the case of the  $CP^N$  model, the  $U(N)$  symmetry can be realized with the collective coordinate ansatz  $Z_\alpha = U_{\alpha\beta} Z_\beta^0$  ( $\alpha = 1, \dots, N$ ) where  $U_{\alpha\beta}$  are the  $N \times N$  unitary matrices and  $Z_\alpha^0$  are the hedgehog solutions such as  $Z_\alpha^0 = 0$  ( $\alpha < N$ ) and  $Z_N^0 = \cos \frac{F(r)}{2}$ . The remnant field  $Z_{N+1}$  cannot be rotated to satisfy the boundary conditions on the gauge invariant physical quantities [6] as in  $Z_2$  in Eq. (4.1) for the  $CP^1$  case.

Using the above soliton configuration, we obtain the unconstrained Lagrangian of the form

$$L = -E + \frac{1}{2} \mathcal{I} \dot{\alpha}^2 + \frac{\Theta}{2\pi} \dot{\alpha}, \quad (4.2)$$

where the soliton static mass and the moment of inertia are given by

$$\begin{aligned} E &= \frac{\pi}{2} \int_0^\infty dr \, r \left[ \left( \frac{dF}{dr} \right)^2 + \frac{\sin^2 F}{r^2} \right], \\ \mathcal{I} &= \pi \int_0^\infty dr \, r \sin^2 F. \end{aligned} \quad (4.3)$$

Introducing the canonical momentum conjugate to the proper collective coordinate  $\alpha$

$$p_\alpha = \mathcal{I}\dot{\alpha} + \frac{\Theta}{2\pi}, \quad (4.4)$$

we then have the canonical Hamiltonian as follows

$$H = E + \frac{1}{2\mathcal{I}}(p_\alpha - \frac{\Theta}{2\pi})^2. \quad (4.5)$$

Here one notes that, at the canonical level, there is an explicit contribution of the Hopf term to this Hamiltonian, which is attainable via the collective coordinate ansatz (4.1).

Then, substituting the configuration (4.1) into Eq. (2.15), we obtain the angular momentum operator of the form

$$J = \mathcal{I}\dot{\alpha} = p_\alpha - \frac{\Theta}{2\pi} = -i\frac{\partial}{\partial\alpha} - \frac{\Theta}{2\pi} = -I - \frac{\Theta}{2\pi}, \quad (4.6)$$

where  $I$  is the isospin quantum number. Here note that the angular momentum  $J$  has the fractional quantum number (integer +  $\frac{\Theta}{2\pi}$ ) where one can see the explicit Hopf term contribution to the total spin. Then, one can obtain the eigenvalues of the Hamiltonian (4.5) as

$$\langle H \rangle = E + \frac{1}{2\mathcal{I}}(I + \frac{\Theta}{2\pi})^2. \quad (4.7)$$

In fact, the above Hamiltonian can be interpreted as mass spectrum of a rigid rotator in the  $CP^1$  model with the Hopf term and, for the case of  $\Theta = \pi$ , the rotator becomes a fermion, as in the (3+1) dimensional Skyrme soliton. Moreover, the zero modes in the extended phase space can be shown to be the same as those in the original phase space with the same energy spectrum (4.7) as in the  $O(3)$  NLSM case [15].

Finally let us define the proper topological charge  $Q$  associated with the global  $U(1)$  symmetry in terms of the canonical momenta  $\Pi_\alpha$  as follows

$$Q = 2i \int d^2x (Z_\alpha \Pi_\alpha - Z_\alpha^* \Pi_\alpha^*) = -\frac{i\Theta}{2\pi^2} \int d^2x \epsilon^{ij} \partial_i Z_\alpha^* \partial_j Z_\alpha, \quad (4.8)$$

which originates only from the Hopf term. This Hopf term plays a crucial role in fermionization of the  $CP^1$  model as the Wess-Zumino-Witten term does in the Skyrme model [20].

Note that, for the case of  $\Theta = 0$ , one can have bosonic description as expected. On the other hand, in the case of  $\Theta = \pi$ , the topological charge  $Q$  can be rewritten as

$$Q = \frac{1}{4\pi} \int d^2x \epsilon^{ij} F_{ij}, \quad (4.9)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  with the definition of the gauge field  $A_\mu$  in Eq. (2.2). Then, the above expression of  $Q$  is exactly the second Chern class associated with a line bundle with U(1)-valued transition function [21,7] and also can be expressed in terms of the topological current as follows

$$Q = \int d^2x B^0, \quad (4.10)$$

where the topological current is given as  $B^\mu = \frac{1}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho}$ . By substituting the collective coordinates (4.1) for the unit soliton sector into Eq. (4.8), one can easily obtain

$$Q = \frac{1}{2} \int_0^\pi dF \sin F, \quad (4.11)$$

which yields unit topological charge and thus explicitly describes the fermion statistics. Moreover, for the general case of  $0 < \Theta < \pi$  in Eq. (4.8), one can show that the topological charge operator  $Q$  has arbitrary fractional spin statistics, which was also seen in the angular momentum operator (4.6).

## V. CONCLUSION

In summary, we have constructed the first-class BFT physical fields, in terms of which the first-class Hamiltonian is formulated to be consistent with the Hamiltonian with the original fields and auxiliary fields. The translational and rotational properties of the physical fields  $Z_\alpha$  have been realized via the Dirac brackets (not the Poisson brackets) of the physical fields. Since we have obtained the first-class Hamiltonian, we have introduced the (anti)ghost fields to obtain the BRST invariant gauge fixed Lagrangian and its BRST transformation rules. On the other hand, introducing the collective coordinates in the soliton configuration, we have performed semiclassical quantization to yield the energy spectrum which can be

interpreted as that of the rigid rotator and can also yield fractional spin statistics via the Hopf term contributions. It will be interesting to study the  $CP^N$  model on the noncommutative geometry through further investigation.

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